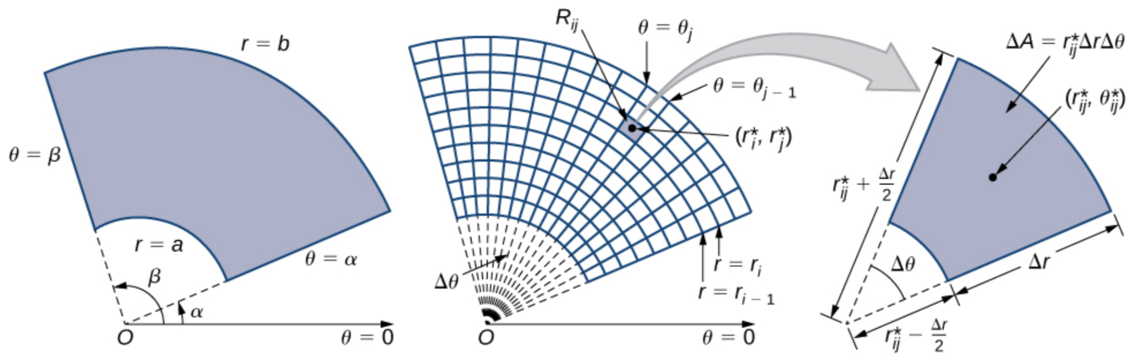
SECTIN 16.3 DOUBLE INTEGRALS USING POLAR COORDINATES

RECALL: POLAR COORDINATES:

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad x^2 + y^2 = r^2, \qquad \tan(\theta) = \frac{y}{x},\ x \neq 0$$

QUESTION: How do we set-up an integral in polar coordinates?



THEOREM: Suppose R is described in polar coordinates as: $R = \{(r, \theta)\ :\ \alpha \le \theta \le \beta,\ g(\theta) \le r \le h(\theta)\}$:

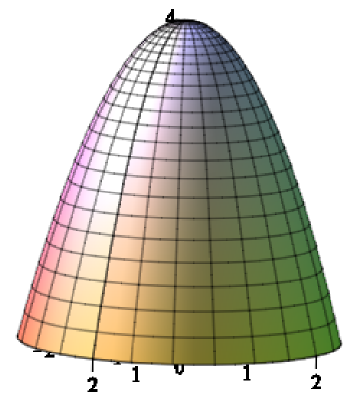
$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

EXAMPLE 1: Convert the following integral from rectangular coordinates to polar coordinates to evaluate.

$$\int_0^1 \int_{x\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$$

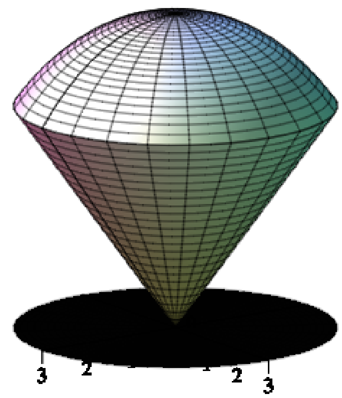
Ans: $\displaystyle\int_0^1 \int_{x\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\, dy\, dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^2 r^2\, dr\, d\theta = \frac{4\pi}{9}$

EXAMPLE 2: Find the volume of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.



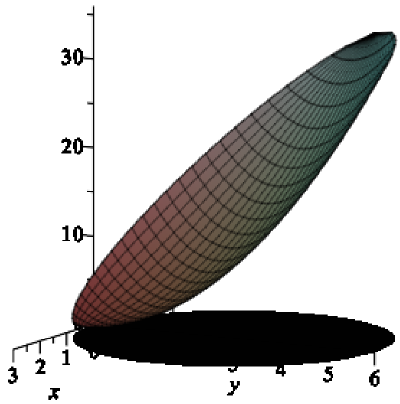
Ans: Volume $= \displaystyle\int_0^{2\pi} \int_0^2 \left(4 - r^2\right) r \, dr \, d\theta = 8\pi \text{ units}^3$

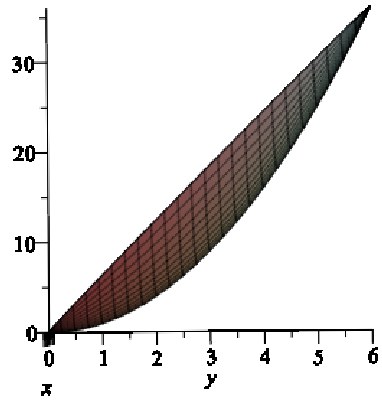
EXAMPLE 3: Find the volume of the solid between the surfaces $z = \sqrt{18 - x^2 - y^2}$ and $z = \sqrt{x^2 + y^2}$.



Ans: Volume $= \int_0^{2\pi} \int_0^3 \left(\sqrt{18 - r^2} - r \right) r \, dr \, d\theta = 36\pi \left(\sqrt{2} - 1 \right)$ units^3

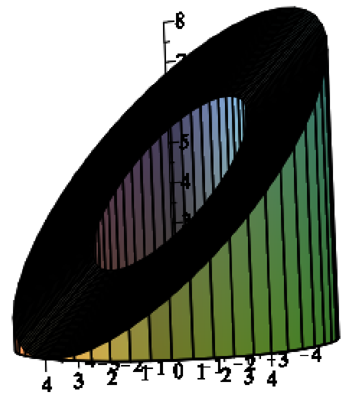
EXAMPLE 4: Find the volume of the solid between the surfaces $z = x^2 + y^2$ and $z = 6y$.





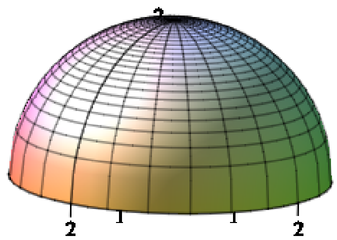
Ans: Volume $= \displaystyle\int_0^{\pi} \int_0^{6\sin(\theta)} \left(6r\sin(\theta) - r^2\right)\, r\, dr\, d\theta = \frac{81\pi}{2}$ units^3

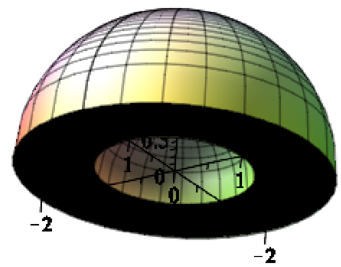
EXAMPLE 5: Set-up an iterated integral in polar coordinates which computes the volume of the solid bounded by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$ and the planes $z = 0$ and $x + z = 4$.



Ans: Volume $= \displaystyle\int_0^{2\pi} \int_2^4 (4 - r\cos(\theta))\ r\,dr\,d\theta = 48\pi$ units^3

EXAMPLE 6: Set-up a sum of iterated integrals in polar coordinates which computes the volume of the solid between the surfaces $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$.

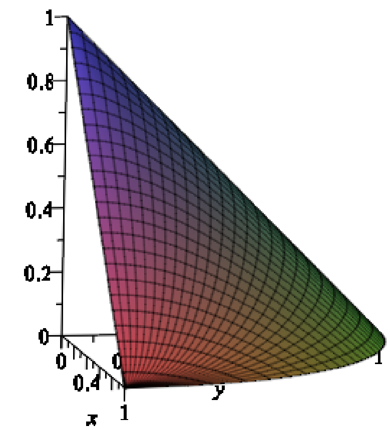




Find the volume using formulas from geometry.

Ans: $\text{Volume} = \displaystyle\int_0^{2\pi}\int_0^1 \left(\sqrt{4-r^2} - \sqrt{1-r^2}\right) r\, dr\, d\theta + \int_0^{2\pi}\int_1^2 r\sqrt{4-r^2}\, dr\, d\theta = \frac{14\pi}{3}\text{ units}^3$

EXAMPLE 7: Write a difference of iterated integrals (one in polar coordinates, one in rectangular coordinates) which computes the volume of the solid bounded by $z = 1 - \sqrt{x^2 + y^2}$, $z = 1 - x - y$, and $z = 0$.



Ans: Volume $= \displaystyle\int_0^{\pi/2} \int_0^1 (1-r)\ r\,dr\,d\theta - \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx = \frac{\pi - 2}{12}$ units^3

EXAMPLE 8: Follow the steps below to find the value of $\displaystyle\int_0^\infty e^{-x^2}\,dx$:

1. Let $N = \displaystyle\int_0^\infty e^{-x^2}\,dx$. Then $N^2 = \left[\displaystyle\int_0^\infty e^{-x^2}\,dx\right]^2 = \left[\displaystyle\int_0^\infty e^{-x^2}\,dx\right]\left[\displaystyle\int_0^\infty e^{-x^2}\,dx\right]$.

2. Why does $\displaystyle\int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-y^2}\,dy$?

3. Hence, $N^2 = \left[\displaystyle\int_0^\infty e^{-x^2}\,dx\right]\left[\displaystyle\int_0^\infty e^{-x^2}\,dx\right] = \left[\displaystyle\int_0^\infty e^{-x^2}\,dx\right]\left[\displaystyle\int_0^\infty e^{-y^2}\,dy\right]$

4. Explain the following manipulation:

$$\left[\int_0^\infty e^{-x^2}\,dx\right]\left[\int_0^\infty e^{-y^2}\,dy\right] = \int_0^\infty e^{-x^2}\left[\int_0^\infty e^{-y^2}\,dy\right]dx = \int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}\,dy\,dx$$

5. Rewrite:

$$\int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}\,dy\,dx = \int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\,dy\,dx$$

Convert $\displaystyle\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\,dy\,dx$ to an integral in polar coordinates and evaluate.

Ans: $\displaystyle\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\,dy\,dx = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta = \frac{\pi}{4}$

6. Use your answer to the previous part to determine the exact value of: $\displaystyle\int_0^\infty e^{-x^2}\,dx$.

HOMEWORK: Section 16.3: 7 - 63 every other odd